Jordan property for Cremona groups (after Prokhorov and Shramov).

$k=\bar{k}$, char $k=0$.

$Cr_n(k) = Bir(\mathbb{P}^n_k)$

<u>Thm</u> (C. Jordan) $\exists I = I(n)$ s.t. $\forall G \subset GL_n(\mathbb{C})$

$\exists A \triangleleft G$ - Abelian s.t. $[G:A] \le I$.

<u>Def</u>: We say that Γ satisfies Jordan property (is Jordan) if $\exists J$ s.t. $\forall G \subset \Gamma, |G|<\infty$ $\exists A \triangleleft \Gamma$, $[\Gamma : A] \le J$.

<u>Example</u>: C - projective curve $\Rightarrow$ $Aut(X)$ is Jordan.

<u>Thm</u> (J.-P. Serre) $Cr_2(\mathbb{K})$ is Jordan.

<u>Thm</u> (Yu. Zarhin) $X \cong E \times \mathbb{P}^1$, E - Abelian of $\dim E > 0$ $\Rightarrow$ $Bir(X)$ is not Jordan.

<u>thm</u> (V. Popov) S - surface. $\Rightarrow$ $Bir(S)$ is Jordan $\Leftrightarrow$ S is not birational to $E \times \mathbb{P}^1$, E - elliptic curve

<u>Question</u> (J.-P. Serre) Is $Cr_n(\mathbb{K})$ Jordan

<u>Thm</u> (Prokhorov–Shramov + Birkar (BAB)): X - rat. conn. of $\dim X = n$ defined over $\mathbb{K}$, char $\mathbb{K} = 0$ $\Rightarrow$ $Bir(X)$ is Jordan

In particular, $Cr_n(\mathbb{K})$ is Jordan

<u>Corollary</u>: X - rat. conn of $\dim = n$, $\exists L = L(n)$ s.t. $\forall G \subset Bir(X)$, $|G| < \infty$ $\forall p > L$ - prime, if G is p-group then G is Abelian generated by at most n elements.

<u>thm</u> (Beauville). In $Cr_2(\mathbb{K})$, G - p-group, $|G| < \infty$ $\Rightarrow$ G can be generated by $\begin{cases} 2, & p \ge 5 \\ 3, & p = 3 \\ 4, & p = 2 \end{cases}$ elements

<u>Thm</u> (Prokhorov–Shramov– Xu – Kuznetsova – L.) X - rat. conn., $\dim = 3$. $G \subset Bir(X)$, G - finite p-group $\Rightarrow$ G can be generated by $\begin{cases} 3, & p \ge 5 \\ 4, & p = 3 \\ 6, & p = 2 \end{cases}$ elements.

Def: $\mathcal{C}$ – set of varieties

$\mathcal{C}$ has almost fixed points property,

if $\exists$ ~~J~~ $\forall G \subset \mathrm{Aut}\, X$, $X \in \mathcal{C}$, $\mathfrak{F}$

$|G| < \infty$ $\exists F \subset G$ of $[G:F] \leq J$

s.t. F has a fixed point on V,

Thm: $\mathcal{R}(n)$ – rat. conn, of dim n

$\Rightarrow \mathcal{R}(n)$ has fixed point property,

Proof: (1) $\mathcal{F}(n)$ – Fano of dim $= n$ terminal $\Rightarrow$ has almost fixed points.

Remark: enough for smooth rat. conn

$X \hookrightarrow \mathbb{P}^{\dim|-mK_X|}$ $\Leftarrow$ BAB

degree $\leq d$,

$G \curvearrowright X$, $\exists$ Abelian $\Gamma \trianglelefteq G$ s.t.

$[G:\Gamma]$ – bounded,

$H_1, \ldots, H_k$ – Γ-inv,

$X \cap H_1 \cap \ldots \cap H_k = \{p_1, \ldots, p_r\}$, $r \leq d$

$H \in \Gamma$, $H = St_{p_i}$, $[G:H]$ – bounded.

(2) works for $\underset{c\text{-MFS}}{f: X \to S}$

Proof: By induction, we may assume that the statement holds for dims $n-1$.

$$1 \to G_f \to G \xrightarrow{\theta} G_S \to 1$$

$\exists\, F_S \subseteq G_S$ of bounded index s.t. F_S has a fixed point p on S.

$F := \theta^{-1}(F_S) \subseteq G$.

$\boxed{\text{Lemma: there exists a rat. conn. } F\text{-inv. variety } Z \text{ in } f^{-1}(p).\ \dim Z < n.}$

$\Rightarrow \exists\, H \subseteq F$ (bounded) s.t. $H \curvearrowright Z$ (and on X) with a fixed point.

~~Proof of Jordan property~~: $G \curvearrowright X$ –

$$\begin{array}{ccc} \text{smooth } G \curvearrowright \widetilde{X} & \xrightarrow{\text{MMP}} & Y \curvearrowleft G \\ \downarrow & & \downarrow \\ X & & Z \end{array}$$

G has almost a fixed point

$\Rightarrow \exists\, H \leq G$ s.t.

$H \curvearrowright T_{x_0}X$

$H \hookrightarrow GL_n(\mathbb{C})$,

$\exists\, A \trianglelefteq H$ of bounded index $\Rightarrow \exists\, A' = \cap A^g$ of bounded index

Lifting of rat conn. varieties.

Prop: $f : V \to W$ – G-cont. V – klt

$D_W \geq 0$ – $\mathbb{Q}$-Cartier, $D = f^* D_W$

$Z \subseteq V$ – minimal non-klt (V, D)

$T = f(Z) \subset W$, $Z_t = Z \cap f^{-1}(t)$

$t \in T$ – general

$\Rightarrow Z_t$ – Fano type.

Prop: $f : V \to W$ – G-contraction, V – klt

$T \subseteq W$ – G-inv. irred

$\exists$ G-inv. irred $Z \subsetneq V$ s.t.

$f|_Z : Z \to T$ – dominant with rat. conn general fiber

Proof: $H_1, \ldots, H_k$ s.t. $Bs\, \mathcal{H} = T$,

$\mathcal{H} = \langle H_1, \ldots, H_k \rangle$

We may assume that $\{H_i\}$ is G-inv

$D_W = \sum H_i$. $D = f^* D_W$.

(V, cD) – lc over general points of T.

S = union of centers that do not dominate T.

Z_1 = center that dominates T.

$Z_1, \ldots, Z_r$ – G-orbit

$W^\circ = W \setminus f(S)$, $Z = \cup Z_i$

$V^\circ = f^{-1}(W^\circ)$

$Z \cap V^\circ$ - minimal G-orbit of non-klt

$(V^\circ, cD|_{V^\circ})$

$\Rightarrow$ general fiber is of Fano type.